Brief overview of classical mechanics

Mechanical systems with conserved energy (i.e. with fixed constraints and no friction):

\[ q, \dot{q} = \frac{dq}{dt} \]

\[ q_1 = x, \dot{q}_1 = \dot{x} \]
\[ q_2 = y, \dot{q}_2 = \dot{y} \]

\[ \mathcal{L} = \mathcal{L}_{q_1, q_2, \ldots, q_N} = \text{"generalized coordinates"} \]

\[ N = \# \text{ of freedoms} \]

Lagrangian mechanics is a way to calculate Newton's laws of motion for the generalized coordinates without calculating the forces of the constraints, i.e. without making free body diagrams.

Basic object is

\[ \mathcal{L}(\dot{q}, q) = T(q, \dot{q}) - V(q) \]

\[ q = \frac{d\gamma}{dt} \text{ treated as independent parameter} \]

\[ \uparrow \text{ kinetic energy} \]

\[ \uparrow \text{ potential energy} \]
Euler-Lagrange equations: \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad , \quad i = 1, \ldots, N \]

Example: Simple pendulum confined to move on a plane

\[ q \rightarrow \theta, \quad v = R \frac{d\theta}{dt}, \quad T = \frac{1}{2}mv^2 \Rightarrow T = \frac{1}{2}mR^2 \dot{\theta}^2 \]

\[ V = mg h = mg R (1 - \cos \theta) \]

\[ L = \frac{1}{2} mR^2 \ddot{\theta}^2 - mgR (1 - \cos \theta) \]

\[ \ddot{\theta} = \frac{d^2\theta}{dt^2} \]

\[ EL: \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow mR^2 \ddot{\theta} + mgR \sin \theta = 0 \Rightarrow \]

\[ \ddot{\theta} + \frac{g}{R} \sin \theta = 0 \quad \text{for small } \theta, \quad \sin \theta \approx \theta \Rightarrow \ddot{\theta} + \frac{g}{R} \theta = 0 \quad \text{harmonic oscillator, } \omega = \sqrt{\frac{g}{R}} \]
Hamiltonian mechanics

\[ p_i = \frac{2L}{q_i} \]

\[ H(\dot{q}_i, \dot{p}_i) = T(\dot{q}_i, \dot{p}_i) + V(q_i) \]

"generalized momentum"

\[ \frac{dq_i}{dt} = \dot{q}_i = \frac{2H}{2p_i} \]

Hamilton's "canonical" equations

\[ \frac{dq_i}{dt} = \dot{q}_i = -\frac{2H}{2q_i} \]

Example: simple pendulum

\[ p_\theta = \frac{2L}{2\dot{\theta}} = mR^2 \dot{\theta} \]

\[ T = \frac{1}{2} mR^2 \dot{\theta}^2 = \frac{p_\theta^2}{2mr^2} \]

\[ V = mgR(1-\cos \theta) \]

\[ H = T + V = \frac{p_\theta^2}{2mr^2} + mgR(1-\cos \theta) \]

\[ \dot{\theta} = \frac{2H}{2p_\theta} = \frac{p_\theta}{mR^2} \]

set of 2 first order

\[ \dot{p}_\theta = -\frac{2H}{2\theta} = mgR \sin \theta \]

note: take time derivative

\[ \dot{\theta} \text{ eqtn} \Rightarrow \ddot{\theta} = \frac{p_\theta}{mr^2} \Rightarrow \ddot{\theta} = \frac{2}{r^2} \sin \theta \]

use \( \dot{p}_\theta \) eqtn same as Euler-Lagrange
Phase space trajectories and oscillatory systems

\[ V(\theta) = mgR(1 - \cos \theta) \]

\[ H = \frac{p_\theta^2}{2mR^2} + mgR(1 - \cos \theta) \]

Turning points

Full phase space portrait

Ellipses for small \( \theta \)

Spendle Phase App: java
Another way to represent motion in phase space:

Pendulum Phase App. java

Hamiltonian flows and Liouville’s theorem: Pendulum PhaseGroupApp.java
Forced oscillations

\[ m \ddot{x} + kx = F_0 \cos \Omega t \implies x = A \cos (\omega t + \phi) + \frac{F_0 \cos \Omega t}{m \left( \omega_0^2 - \Omega^2 \right)} \]

\( \omega = \sqrt{\frac{k}{m}} \)

diverges at \( \omega = \frac{\Omega}{2} \) "resonance"

solution of homogeneous equation

particular solution

\( A, \phi \) determined by initial conditions

Forced damped oscillations

\[ m \ddot{x} + 2m \beta \dot{x} + kx = F_0 \cos \Omega t \]

\[ x = A \cos (\omega t + \phi) + \Gamma \cos \left( \Omega t + \phi \right) \]

\[ \Gamma = \frac{F_0 / m}{\sqrt{\left( \omega_0^2 - \Omega^2 \right)^2 + 4 \beta^2 \Omega^2}} \]

\( \tan \gamma = \frac{2 \beta \Omega}{\left( \omega_0^2 - \Omega^2 \right)} \)

\( \gamma = \frac{\pi}{2} \) at \( \omega_0 \)

\( \omega_0 \)